A Theorem respecting the Singularities of Curves of Multiple Curvature.

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In a paper which appeared in Vol. VIII, No. 2, of this Journal, I showed that any singularity, however complex, possible to an element of a point curve of double curvature may be completely defined by three singularity indices, $\varkappa_1, \varkappa_2, \varkappa_3$. If A_0 be the element and $P_t = A_0 + A_1 t + A_2 t^2 +$ etc. be the Grassmann equation of the curve for its neighborhood, $\varkappa_1$ is the number of consecutive coefficients A_1, A_2, etc., congruent with A_0; $\varkappa_2$, the number of consecutive coefficients $A_{\kappa_1+2}, \ldots.$ linearly derivable from A_0 and A_{κ_1+1}; $\varkappa_3$, the number of consecutive coefficients $A_{\kappa_1+\kappa_2+3} \ldots.$ linearly derivable from $A_0, A_{\kappa_1+1}, A_{\kappa_1+\kappa_2+2}$. The point P_{dt} is then distant an infinitesimal of the order $\varkappa_1 + 1$ from A_0, an infinitesimal of the order $\varkappa_1 + \varkappa_2 + 2$ from $A_0 A_{\kappa_1+1}$, the tangent at A_0, and an infinitesimal of the order $\varkappa_1 + \varkappa_2 + \varkappa_3 + 3$ from $A_0 A_{\kappa_1+1}, A_{\kappa_1+\kappa_2+2}$, the osculating plane at A_0. In other words, the point A_0 of the singularity indices $\varkappa_1, \varkappa_2, \varkappa_3$ is stationary to the degree $\varkappa_1$, a point of contact of the order $\varkappa_1 + \varkappa_2 + 1$ of the curve with its tangent, and a point of contact of the order $\varkappa_1 + \varkappa_2 + \varkappa_3 + 1$ of the curve with its osculating plane.

The singularities of curves of any degree of curvature whatsoever admit of similar treatment. Besides the singularities of which $\varkappa_1, \varkappa_2, \varkappa_3$ measure the degree, and which may be conveniently called singularities of the 1st, 2d and 3d class respectively, there will be possible to a curve of $(n-1)^{\text{ple}}$ curvature singularities of a 4th, 5th, ... nth class also, in consequence of the n dimensionality of the space supposed. Let indices $\varkappa_4 \ldots. \varkappa_n$ measure the degrees of these additional singularities, and $\varkappa_1 \ldots. \varkappa_n$ will completely define the singularity of any curve point A_0. $\varkappa_4$ represents the number of consecutive coefficients $A_{\kappa_1+\kappa_2+\kappa_3+4} \ldots.$ in the Grassmann equation, which are linearly derivable from $A_0, A_{\kappa_1+1}, A_{\kappa_1+\kappa_2+2}, A_{\kappa_1+\kappa_2+\kappa_3+3}$; and $\varkappa_5$, etc., have analogous meanings.

A curve of $(n-1)^{\text{ple}}$ curvature will be touched at every point, not only by a definite line and plane, but, when $n > 3$, by a definite space of 3 dimensions, one of 4 dimensions, and so on to one of $n-1$ dimensions also; the tangent S_3 at any curve point A_0, for instance, being the S_3 determined by any four curve points P_1, P_2, P_3, P_4 in what is its limiting position when P_1, P_2, P_3, P_4 are made, independently of each other, to approach A_0. And the point A_0 of singularity indices $\varkappa_1 \ldots \varkappa_n$, besides being stationary to the degree $\varkappa_1$, a point of contact with the tangent of the order $\varkappa_1 + \varkappa_2 + 1$, and with the osculating plane of the order $\varkappa_1 + \varkappa_2 + \varkappa_3 + 1$, is a point of contact of the order $\varkappa_1 + \varkappa_2 + \varkappa_3 + \varkappa_4 + 1$ with the osculating S_3, etc.

Again, if the system of osculating planes be taken as the elements of a curve of double curvature, one of these may be singular. This singularity, like the singularity of a point, is capable of definition by three indices. The first of these represents the number of consecutive coefficients congruent with the singular element in the tangential equation of the curve for its neighborhood, and the other two have the same sort of correspondence to $\varkappa_2$ and $\varkappa_3$.

In the paper already referred to, it was shown that the singularity indices of a plane element of a curve of double curvature corresponding to a point element of the singularity indices $\varkappa_1, \varkappa_2, \varkappa_3$ are $\varkappa_3, \varkappa_2, \varkappa_1$. I wish now to prove, for a curve of any degree of curvature, $n-1$, that $\varkappa_n, \varkappa_{n-1}, \ldots \varkappa_1$ *are the singularity indices of the osculating* S_{n-1} *which corresponds to a point of the singularity indices* $\varkappa_1, \varkappa_2, \ldots \varkappa_n$.

Let the equation $P_t = A_0 + A_1 t + A_2 t^2 +$ etc., referred hitherto to a system of coordinates consisting of any set of $n+1$ linearly independent points, be transformed to the system $E, e_1, e_2, \ldots e_n$, of which E is A_0 and $e_1, e_2, \ldots e_n$ are unit vectors each at right angles to all the rest and so directed that Ee_1 is the tangent line to the curve at A_0, Ee_1e_2 the tangent $S_2, \ldots Ee_1e_2 \ldots e_{n-1}$ the tangent S_{n-1}. It will become

$$P_t = E + (\alpha t^{\alpha_1} + \text{etc.})\, e_1 + (\beta t^{\alpha_2} + \text{etc.})\, e_2 + \text{etc.} \ldots + (\nu t^{\alpha_n} + \text{etc.})\, e_n,$$

where

$$\begin{aligned} \alpha_1 &= \varkappa_1 + 1, \\ \alpha_2 &= \varkappa_1 + \varkappa_2 + 2, \\ &\ldots\ldots\ldots\ldots \\ \alpha_n &= \sum_1^n \varkappa_i + n. \end{aligned}$$

The thing to be especially noticed is that the sufficient and necessary conditions of point singularities of the various classes do, when the curve is referred

to the system of coordinates described, find full expression in $\alpha_1, \alpha_2, \ldots \alpha_n$, the exponents of the lowest powers of t in the coefficients of $e_1, e_2, \ldots e_n$ respectively, in this equation; $\alpha_1 - 1$ being the index of singularity of class 1, $\alpha_2 - \alpha_1 - 1$ the index of singularity of class 2, etc.

To demonstrate our theorem, therefore, it is only necessary to show that in the tangential equation of the curve for the neighborhood of the osculating S_{n-1} at A_0 the successive exponents corresponding to $0, \alpha_1, \alpha_2$, etc., are $0, \alpha_n - \alpha_{n-1}, \alpha_n - \alpha_{n-2}, \ldots \alpha_n - \alpha_1, \alpha_n$; since, by a simple retracing of the above reasoning, the index of singularity of class 1 of this osculating S_{n-1} (we may call it ε_0) would then be $\alpha_n - \alpha_{n-1} - 1$, *i. e.* $\varkappa_n$; of class 2, $(\alpha_n - \alpha_{n-2}) - (\alpha_n - \alpha_{n-1}) - 1$, *i. e.* $\varkappa_{n-1} \ldots$; finally, of class n, $\alpha_n - (\alpha_n - \alpha_1) - 1$, *i. e.* $\varkappa_1$.

But the tangential equation of the curve for the neighborhood of ε_0 is the product $P_t \cdot \frac{d^{\alpha_1} P_t}{dt^{\alpha_1}} \cdot \frac{d^{\alpha_2} P_t}{dt^{\alpha_2}} \ldots \frac{d^{\alpha_{n-1}} P_t}{dt^{\alpha_{n-1}}}$, as is easily inferred from the definition of a tangent S_{n-1}, by getting the limit, as dt approaches 0, of the product $P_{t+\rho_1 dt} \cdot P_{t+\rho_2 dt} \ldots P_{t+\rho_n dt}$. Expanding, we have

$$\begin{aligned}\varepsilon_t &= P_t \cdot \frac{d^{\alpha_1} P_t}{dt^{\alpha_1}} \cdot \frac{d^{\alpha_2} P_t}{dt^{\alpha_2}} \ldots \frac{d^{\alpha_{n-1}} P_t}{dt^{\alpha_{n-1}}} \\ &= f_{(n)}(t) E e_1 e_2 \ldots e_{n-1} + f_{(n-1)}(t) E e_1 e_1 \ldots e_{n-2} e_n + \ldots f_{(0)}(t) e_1 e_2 \ldots e_n,\end{aligned}$$

any coefficient $f_{(n-j)}(t)$ being the determinant of the coefficients of

$$E, e_1, e_2, \ldots e_{n-j-1}, e_{n-j+1}, \ldots e_n \text{ in } P_t, \frac{d^{\alpha_1} P_t}{dt^{\alpha_1}}, \text{ etc.}$$

As we are concerned only to determine what is the lowest power of t in each of these determinants, we write, instead of the elements themselves, the exponent of the lowest power of t in each element, and, after a simple reduction, have for $f_{(n-j)}(t)$,

$$\left|\begin{array}{ccccc|ccc}
0 & \alpha_2 - \alpha_1 & \alpha_3 - \alpha_1 & \ldots & \alpha_{j-1} - \alpha_1 & \alpha_{j+1} - \alpha_1 & \ldots & \alpha_n - \alpha_1 \\
0 & 0 & \alpha_3 - \alpha_2 & \ldots & \alpha_{j-1} - \alpha_2 & \alpha_{j+1} - \alpha_2 & \ldots & \alpha_n - \alpha_2 \\
0 & 0 & 0 & \ldots & \alpha_{j-1} - \alpha_3 & \alpha_{j+1} - \alpha_3 & \ldots & \alpha_n - \alpha_3 \\
\cdot & \cdot & \cdot & \ldots & \cdot & \cdot & \ldots & \cdot \\
\cdot & \cdot & \cdot & \ldots & \cdot & \cdot & \ldots & \cdot \\
\cdot & \cdot & \cdot & \ldots & \alpha_{j-1} - \alpha_{j-2} & & & \\
0 & 0 & 0 & \ldots & 0 & \alpha_{j+1} - \alpha_{j-1} & \ldots & \alpha_n - \alpha_{j-1} \\
\hline
0 & 0 & 0 & \ldots & 0 & \alpha_{j+1} - \alpha_j & \ldots & \alpha_n - \alpha_j \\
0 & 0 & 0 & \ldots & 0 & 0 & \ldots & \alpha_n - \alpha_{j+1} \\
\cdot & \cdot & \cdot & \ldots & \cdot & \cdot & \ldots & \cdot \\
\cdot & \cdot & \cdot & \ldots & \cdot & \cdot & \ldots & \cdot \\
0 & 0 & 0 & \ldots & \cdot & 0 & \ldots & \alpha_n - \alpha_{n-1}
\end{array}\right| .$$

The lowest power of t in $f_{(n-j)}(t)$ is then the smallest of the sums, each of n elements, of the above array so taken that no two are in the same row or column; and this smallest sum is $\alpha_n - \alpha_j$, as the following considerations show.

It is indeed necessary only to demonstrate that $\alpha_n - \alpha_j$ is the smallest of the sums built in the manner described out of the elements of the minor $\ulcorner$; for then $\alpha_n - \alpha_j$, plus the smallest sum of $\lrcorner$, is $\alpha_n - \alpha_j$, while complete sums which do not involve the sums of $\llcorner$ must involve one of the numbers $\alpha_n - \alpha_1, \alpha_n - \alpha_2, \ldots,$ $\alpha_n - \alpha_{j-1}$, and any of these is itself greater than $\alpha_n - \alpha_j$. For convenience, we substitute a, b, c, etc., for α_j, α_{j+1}, etc., and write $\ulcorner$ in the form

$$\begin{vmatrix} b-a & c-a & d-a & e-a & \ldots & m-a \\ 0 & c-b & d-b & e-b & \ldots & m-b \\ 0 & 0 & d-c & e-c & \ldots & m-c \\ 0 & 0 & 0 & e-d & \ldots & m-d \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & 0 & \ldots & m-l \end{vmatrix}.$$

It is to be noted that any element is less than that following it in the same row, greater than that following it in the same column. The least sum in

$$\begin{vmatrix} b-a & c-a & d-a \\ 0 & c-b & d-b \\ 0 & 0 & d-c \end{vmatrix}$$

is obviously $d-a$. But from this it follows that the least sum in

$$\begin{vmatrix} b-a & c-a & d-a & e-a \\ 0 & c-b & d-b & e-b \\ 0 & 0 & d-c & e-c \\ 0 & 0 & 0 & e-d \end{vmatrix}$$

is $e-a$; for $e-d+$ the least sum in the minor of $e-d$ is the least sum in which $e-d$ occurs, and $e-d+d-a=e-a$; $e-c+$ the least sum in the minor got by striking out the last two rows and columns of the determinant is the least in which $e-c$ occurs, since this minor involves lower elements than any other that can be constructed out of the first two rows of the determinant, and $e-c+c-a=e-a$; and $e-b+b-a$, the minor got by canceling the last three rows and columns of the determinant, is the least in which

$e - b$ occurs, and is again $e - a$. And, by the same method of proof, it follows that since the theorem holds for determinants of the orders $1 - 4$, it holds for one of the order 5, etc.

The exponent of the lowest power of t in the coefficient of any term, $Ee_1e_2 \dots e_{j-1} \cdot e_{j+1} \dots e_n$, is therefore $\alpha_n - \alpha_j$, as was to be proved, that of $e_1e_2 \dots e_n$ in particular being α_n. Calling $Ee_1e_2 \dots e_{j-1}e_{j+1} \dots e_n$, ε_j, we have, as the tangential equation of the curve,

$$\varepsilon_t = \varepsilon_0 + (\alpha' t^{\alpha_n - \alpha_{n-1}} + \text{etc.})\,\varepsilon_1 + (\beta' t^{\alpha_n - \alpha_{n-2}} + \text{etc.})\,\varepsilon_2 + \text{etc.}, \dots (\nu' t^{\alpha_n} + \text{etc.})\,\varepsilon_n .$$

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